Problem Solving Strategies 3

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Invariance Principle $\longrightarrow$ This is a heuristic principle

Here we study the strategy to find something that does not vary.

- We are starting from a point (a, b) of the plane with $0 < b < a$.
We make a sequence (x_n, y_n) such that,

$$x_0 = a, \; y_0 = b \qquad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

Here we can define the invariant will be,

$$x_{n+1} y_{n+1} = x_n y_n \text{ for all values of } n \geq 0$$

$$\Rightarrow x_{n+1} y_{n+1} = x_n y_n = x_{n-1} y_{n-1} \cdots = x_0 y_0 = ab$$

$$\Rightarrow x_n y_n = ab \;\forall\, n \geq 0$$ — this is also an invariant

$y_0 < x_0$

Let us suppose $y_n < x_n$ for some n

y_{n+1} is the HM of x_n and y_n and x_{n+1} is the AM of x_n and y_n

$$\Rightarrow y_{n+1} < x_{n+1}$$

So this relation $y_n < x_n$ is also an invariant

$$0 < x_{n+1} - y_{n+1} < \underbrace{\frac{x_n - y_n}{x_n + y_n}}_{<1} \cdot \frac{x_n - y_n}{2} < \frac{x_n - y_n}{2} < \frac{x_{n-1} - y_{n-1}}{2^2} \cdots < \frac{x_0 - y_0}{2^{n+1}}$$

$$\lim_{n\to\infty} (x_n - y_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \sqrt{ab}$$ .... as AM $\geq$ GM $\geq$ HM

So if AM = HM $\Rightarrow$ AM = GM = HM

Now we have the

Q> Let us have a positive integer n which is odd. Now we have the numbers $1, 2, \ldots, 2n$ written is board. We pick any two numbers a, b, then erases them and write $|a-b|$ instead. Prove that an odd number will remain in the board at the end.

Ans:- $S = 1+2+3+\cdots+2n = \frac{2n(2n+1)}{2} = n(2n+1)$ which is odd

S is odd first.

After choosing a, b and putting $|a-b|$ we get $S-a-b+|a-b|$ as the new sum.

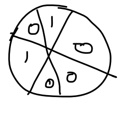
WLOG $b<a$, then $|a-b| = a-b$

$$S-a-b+a-b = S-2b$$ is odd again as S is odd and $2b$ is even.

So we get that summation of the remaining numbers is odd and invariant

$\Rightarrow$ The last remaining numbers is also odd

Q> A circle is divided into six sectors. Then the numbers $1, 0, 1, 0, 0, 0$ are written into the sectors (counterclockwise). Now we can increase the two neighbouring numbers by 1. Is it possible to equalize all the numbers by such steps.



Ans:- $\overset{=1}{a_1}, \overset{=0}{a_2}, \overset{=1}{a_3}, \overset{=0}{a_4}, \overset{=0}{a_5}, \overset{=0}{a_6}$ initially be the numbers

Now we define $S = a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

After each step S remains same as we get $+1$ and -1 which is the invariant.

If $a_1 = a_2 = a_3 = a_4 = a_5 = a_6$ then $S = 0$ but initially $S = 2$. so not possible

Q> There is a group of people where each member has at most three enemies. Prove that they can be separated into two groups so that each members has at most one enemy in his own group.

Ans:– Let us divide them into two groups G_1 and G_2.

Now if A has at least two enemies in his own group

$\Rightarrow$ A has at most one enemy in the other group

So we will shift A to the other group

So the total number of enemy relations in the own groups (say E) decrease.

So E can be decreased by at least 1 after each shift.

Now E was finite and so this decrease will terminate.

So after a finite number of steps there will be no such A

Hence proved.

$\Rightarrow$ Here in the last problem the invariant was the decrease in value of E.

HomeWork

Q) We have four integers a, b, c, d and not all four are equal. Then we start with (a, b, c, d) and in each step we replace it by (a−b, b−c, c−d, d−a). Then show that at least one of the numbers in the quadruple will become arbitrarily large

Hint:- Find $a_n^2+b_n^2+c_n^2+d_n^2$.

Sol:- $a_n+b_n+c_n+d_n = 0 \quad \forall\, n \geq 1 \longrightarrow$ One invariant

$$a_{n+1}^2+b_{n+1}^2+c_{n+1}^2+d_{n+1}^2 = (a_n-b_n)^2+(b_n-c_n)^2+(c_n-d_n)^2+(d_n-a_n)^2$$
$$= 2(a_n^2+b_n^2+c_n^2+d_n^2) - 2(a_nb_n+b_nc_n+c_nd_n+d_na_n) \quad — (1)$$

$$0 = (a_n+b_n+c_n+d_n)^2 = (a_n+c_n)^2+(b_n+d_n)^2 + 2a_nb_n+2b_nc_n+2c_nd_n+2d_na_n \quad — (2)$$

(1) + (2)

$$\Rightarrow a_{n+1}^2+b_{n+1}^2+c_{n+1}^2+d_{n+1}^2$$
$$= 2(a_n^2+b_n^2+c_n^2+d_n^2) + (a_n+c_n)^2+(b_n+d_n)^2$$

$= 2(a_n \ldots$

$\Rightarrow a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2 \geq 2(a_n^2 + b_n^2 + c_n^2 + d_n^2)$

$\Rightarrow a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2 \geq 2^n(a_0^2 + b_0^2 + c_0^2 + d_0^2)$

$\Rightarrow$ So one of a_n, b_n, c_n, d_n must be arbitrarily large as $n \to \infty$

$a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2 \to \infty$